# Practice Exam Stochastic Processes

June 2020

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- You have from 15.00 until 18.30. This includes the time needed to take pictures of your work and upload it to nestor dropbox.
- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.

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**Exercise 1 (20 pts).**
Consider the one-dimensional symmetric random walk ($S_n = X_1 + \cdots + X_n$ with the $X_i$ i.i.d. with $\mathbb{P}(X_1 = -1) = \mathbb{P}(X_1 = +1) = \frac{1}{2}$). Show that, for each $n, x \in \mathbb{N}$:

$$\mathbb{P}(S_n \geq x) \leq \mathbb{P}\left(\max_{1 \leq m \leq n} S_m \geq x\right) \leq 2\mathbb{P}(S_n \geq x).$$

**Exercise 2 (20 pts)**
Show that, for every $\mu > 1$ and $0 < q < 1$ there exists an $X$ such that $\mathbb{E}X = \mu$ and the Galton-Watson process with offspring distribution $X$ has probability of extinction exactly equal to $q$.

**Exercise 3 (20 pts)**
State and prove Thompson's principle.

**Exercise 4 (20 pts)**
Consider a Markov chain whose transition matrix $P$ is symmetric (i.e. $P_{ij} = P_{ji}$ for all $i, j$). Show that the uniform distribution is stationary.

**Exercise 5 (20 pts)** Consider a *biased random walk*, i.e. $S_n = X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are i.i.d. with $\mathbb{P}(X_1 = +1) = p, \mathbb{P}(X_1 = -1) = 1 - p$. Supposing $p < \frac{1}{2}$ show that for all $r \in \mathbb{N}$:

$$\mathbb{P}(\max_n S_n \geq r) = \left(\frac{p}{1-p}\right)^r.$$

**the end**